

Harmonic maps

- Refs:
- Fung-Hua Lin & Chang-Yen Wan: HM's and their heat flows.
 - Schoen & Wittenberg: a seg. theory for HM's. (typo)
 - F. Hélein: HM's, conservation laws, & moving frames.
 - J. Jost "Riem. geom. & geom. anal."
 - L. Simon "Theorems on reg. d.s. of minimizing HM's."
 - M. Spence "Variational methods."

Trying to cover too

(M, g) dim. m $\rightsquigarrow$ $\{2\}$ many
 (N, h) , opt., dim. n $\rightsquigarrow$ $\{1\}$ boxes.

$$u: M \rightarrow N \Rightarrow du = \left(\frac{\partial u^i}{\partial x^j} \right) = (u_j^i)$$

$$e(u) = \frac{1}{2} g^{\alpha\beta} h_{ij} u_j^\alpha u_i^\beta = \frac{1}{2} |du|^2$$

$$E(u) = \int_M e(u) dV_g - \text{Dirichlet energy.}$$

Defn: $u \in C^2(M, N)$ is a harmonic map

$\Leftrightarrow$ crit. pt. of $E(u)$, i.e. $\forall u_t: D_t \times [-\epsilon, \epsilon] \rightarrow N$

$$u|_{t=0} = u, \quad \frac{d}{dt} E(u_t) \Big|_{t=0} = 0.$$

Extrinsic viewpoint:

$$N \subset \mathbb{R}^L \quad (\text{Nash})$$

$$C^2(M, N) = \{u: M \rightarrow \mathbb{R}^L \mid u(M) \subset N\}$$

$$e(u) = \frac{1}{2} g^{\alpha\beta} u_\alpha^i u_\beta^i \quad \text{for opt.}$$

Prop: $u: M \rightarrow N$ is harmonic $\Leftrightarrow (\Delta u)^T = 0$.

i.e. $\Delta u \perp T_u N$

Pf: N qpt $\Rightarrow \exists \delta > 0$ s.t. $\Pi: N_\delta \rightarrow N$ smooth.

Write $\phi|_g = d_g \Pi: \mathbb{R}^L \rightarrow T_g N$.

Let $\phi: M \rightarrow \mathbb{R}^L$ qpt. support.

$$0 = \frac{d}{dt} \bigg|_{t=0} \frac{1}{2} \int |\nabla(\Pi(u + t\phi))|^2 dV_g$$

$$= \int \langle \nabla u, \nabla(P(u)(\phi)) \rangle dV_g$$

$$= - \int \langle \Delta_g u, P(u)(\phi) \rangle dV_g$$

$$= - \int \langle P(u)(\Delta_g u), \phi \rangle dV_g = 0$$

$$\Rightarrow P(u)(\Delta_g u) = 0.$$

□

Chromatic viewpoint:

Can't be switching back & forth

$$du \in T(T^*M \otimes u^*TN)$$

$$g \otimes u^*h$$

metric

$$u^*T + u^*(T)$$

comput.

$$u^*(T) = \frac{\partial u^i}{\partial x^j} \frac{\partial}{\partial x^j}$$

connection

Define $T(u) = \text{tag } \nabla du$, tension field.

$$T^k(u) = g^{\alpha\beta} \left(u_{\alpha\beta}^k - \Gamma_{\alpha\beta}^\gamma u_\gamma^k + u_{ij}^k u_\alpha^i u_\beta^j \right)$$

$$T^k(u) = \Delta_g u^k + u_{ij}^k u_\alpha^i u_\beta^j g^{\alpha\beta}$$

Claim: $HM \Leftrightarrow T(\mathcal{H}) = 0$.

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Exercise: Let $A(y) = \nabla P^+(y) : T_y M \otimes T_y M \rightarrow T_y N$ and f.f.

Lemma: $u: M \rightarrow N$ any map.

$$\text{Then } (\Delta_g u)^\perp = -A(u)(\nabla u, \nabla u) g^{\alpha\beta}.$$

Pf: $\Delta_g u = \nabla_j \nabla^j u = 0$.

$$\Rightarrow \nabla_j (\nabla^j u) = 0 \Rightarrow \nabla_j (\nabla^j u) = 0$$

$$\Rightarrow \nabla_j (\nabla^j u) = -u^i \nabla_j \nabla^j P^+(u) = -A(u)(\nabla u, \nabla u) g^{\alpha\beta}.$$

Cor: $(\Delta_g u)^\perp \perp T_u N$

$$\Leftrightarrow \Delta_g u + A(u)(\nabla u, \nabla u) = 0.$$

Example: $M = \mathbb{R}^n$, $N = S^n \subset \mathbb{R}^{n+1}$.

$$[HM \Leftrightarrow \Delta u + |\nabla u|^2 u = 0.]$$

→ just state the outward viewpoint here! cuz it comes directly from this.

Basic facts abt HM:

(a) $S' \rightarrow N$ is HM $\Leftrightarrow$ closed geodesic.

(b) $g \rightsquigarrow e^{2v} g = \tilde{g}$

$$|\nabla u|_{\tilde{g}} \equiv e^{-2v} |\nabla u|_g, \quad dV_{\tilde{g}} = e^{mv} dV_g$$

$$\begin{aligned} \Rightarrow E_{\tilde{g}}(u) &= \frac{1}{2} \int |\nabla u|_{\tilde{g}}^2 dV_{\tilde{g}} \\ &= \frac{1}{2} \int |\nabla u|_g^2 e^{(m-2)v} dV_g \end{aligned}$$

$m=2 \Rightarrow$ Dirichlet en. is conf. inv!

$\Rightarrow$ set of HM's $M \rightarrow N$ only depends on the conformal class $[g]$.

(c) identity map is harmonic.

(d) $m=n=2$. Conformal maps are harmonic.

(e) $m=2$, n arb.

u is conformal & harmonic $\Leftrightarrow$ parametric minimal surface

(f) $M = S^2$: harmonic $\Rightarrow$ conformal (Hart. d. H!)
 harmonic $\Leftrightarrow$ parametric min'l surf.

Ex: $u: S^2 \rightarrow S$ HM's $\Leftrightarrow$ Möbius transfs.

Bochner f'la for Hdr's:

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$$\Delta_g e(m) = |\nabla d_m|^2 + R_{\alpha\beta}(m, m) - {}^N R_{ijkl} n_i^j n_k^l$$

P1: $\nabla \rightarrow$ ordered metric on $T^*M \otimes T^*M$.

Calculate $(\Delta d_m)_\beta = \nabla_\alpha \nabla_\beta d_m^\alpha$

Should maybe have just said this just b/c ∇^2, ∇ for free

note: $\nabla_\beta d_m^\alpha = \nabla_\beta n^\alpha$

$$\Rightarrow \Delta(d_m)_\beta = \nabla_\alpha \nabla_\beta n^\alpha$$

$$= \nabla_\beta \nabla_\alpha n^\alpha + [\nabla_\alpha, \nabla_\beta] n^\alpha$$

$$= \cancel{\nabla_\beta e(m)} - {}^N R_{\alpha\beta} n^\alpha + {}^N R_{jkl}^i n_i^j n_k^l n_\beta^l$$

$$\Delta_g e(m) = \langle \Delta d_m, d_m \rangle + |\nabla d_m|^2$$

$$= |\nabla d_m|^2 + R_{\alpha\beta} n_i^\alpha n_j^\beta - {}^N R_{ijkl} n_i^j n_k^l n_\beta^l$$

Con: (m.g) opt, $Ric_m \geq 0$, $Sec_m \leq 0$.

any HM has $\nabla d_m = 0$, i.e. totally geodesic.

Or: Should have done the Bochner f'la, ^{properly} ~~drawn~~ ~~stated~~ an a priori bound, and stated

Gells - Sampson. (maybe also Sachs - Uhlen.) and con's.

(Could tell the luggers the defs of weak rds later! garden's time to redress history (of just not ~~knowing~~ getting the point of the ^{extremal} ~~interesting~~) in just one talk.)

Q. Given $m_0: M \rightarrow N$, does there exist a harmonic map m in the homotopy class $[m_0]$?

Prop: (Eells - Wood.) There is no H.M.
 $T^2 \rightarrow S^2$ of degree 1.

Prop: (Lemaire) For $B_r \subset \mathbb{R}^m$, if $u: B_r \rightarrow N$
 is a smooth H.M. w/ $u|_{\partial B_r} = \text{const.}$, then $u = \text{const.}$ (m=2 for any dimension.)

Theorem (Eells - Sampson) Assume M is ypt
 and $\text{Sel}_N \leq 0$. Then any map $m_0: M \rightarrow N$
 homotopic to a smooth harmonic map.
(Sacks - Uhlenbeck)

Theorem: (i) Assume $m=2$, i.e. M is a
 ypt Riemann surface, and $\pi_2(N) = 0$.

Then any map $m_0: M \rightarrow N$ is homotopic to
 a H.M.

(ii) If $\pi_2(N) \neq 0$, it is generated by nontrivial
 minimal spheres. (Better than Almgren - Pitts!)
(Think of Medusa.)

Regularity of weak sh.

$$H(M, N) = \{ u: M \rightarrow \mathbb{R}^n \mid u \in H^1 \text{ and } u(x) \in N \text{ a.e. x} \}$$

Def: u is a weak H.M. if $\Delta u \perp T_x N$ weakly.
(can go back to def.)

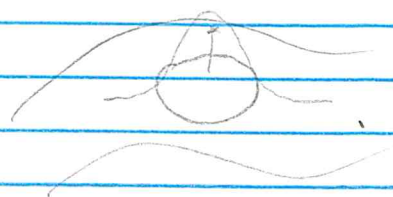
Theorem (Helin, '90s) $n=2 \Rightarrow$ any weakly H.M. is smooth.

Example (Dirac): $\exists$ weakly H.M. $u: B^3 \rightarrow S^2$ which is everywhere discontinuous.

Need stronger notions.

Def: $u: M \rightarrow N$ is a minimizing H.M. if for each $\text{cdt } B \ni x$, $B \subset \text{cdt } s.t.$ for any $v \in H^1$ with $u=v$ on $M \setminus B$, $E(v) \geq E(u)$.

Theorem (Morrey, '48). $n=2$: any min. H.M. is smooth.



Example: $n \geq 3$: $u = \frac{x}{|x|}: B^n \rightarrow S^{n-1}$

is a minimizing H.M. ~~cannot be approx.~~

Cannot be approx. in H^1 by smooth maps $M \rightarrow N$.
($n=3$: cannot be approximated in H^1 .)
by smooth maps.

Theorem (Schoen-Uhlenbeck). The singular set S of a min. HM has Hausdorff codim(S) ≥ 3 , and is discrete if $m=3$. If N contains no weakly harmonic S^l 's for $2 \leq l \leq k$, then codim(S) $\geq k+2$.

$\{\text{minimizing}\} \subset \{\text{stable + stationary}\} \subset \{\text{stationary}\} \subset \{\text{weakly Har}\}$
 $\downarrow$ $\downarrow$
 2nd var. ≥ 0 . also crit for $\Phi_1: M \rightarrow M$
 $(N=S^k)$. $m_+ = m \circ \Phi_+$